HFVS: An arbitrary high order approach based on flux vector splitting

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ABSTRACT

In this paper, a new scheme of arbitrary high order accuracy in both space and time is proposed to solve hyperbolic conservative laws. The basic idea in the construction is that, based on the idea of the flux vector splitting (FVS), we split all the spatial and time derivatives in the Taylor expansion of the numerical flux into two parts: one part with positive eigenvalues, another with negative eigenvalues. According to a Lax-Wendroff procedure, all the time derivatives are then replaced by spatial derivatives, which are evaluated by using WENO reconstruction polynomials. One of the most significant advantages of the current scheme is very easy to implement. In addition, it is found that the higher spatial and time derivatives produced in the construction of the numerical flux can be regarded as a building block, in the sense that they can be coupled with any exact/approximate Riemann solvers to extend a first-order scheme to very high order accuracy in both space and time. Numerous numerical tests for linear and nonlinear hyperbolic conservative laws are carried out, and the numerical results demonstrate that the proposed scheme is robust and can be of high order accuracy in both space and time.
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1. Introduction

In the recent years, numerical schemes of high order accuracy for hyperbolic conservative laws have attracted much attention. In the 1980s, thanks to the efforts of many researchers, for example, Harten [14], Van Leer [44], Roe [23], Sweby [31], et al., schemes of second order accuracy tended to maturity. Later, ENO/WENO schemes show a possibility to construct the schemes of arbitrary high order accuracy in space [13,15]. And since the end of the 1990s, designing schemes of high order accuracy (greater than second order accuracy) becomes one of the most important issues in CFD. In practical simulations, ENO/WENO schemes usually couple with the multi-stage Runge-Kutta method for time evolution. In implementation, the third order TVD Runge-Kutta method is often used [1], since higher than third order Runge-Kutta methods will become complicated in implementation. For instance, the fourth order TVD Runge-Kutta method needs to save all the variables in intermediate stages, which will definitely introduce more computational complexity. Moreover, higher than fourth order Runge-Kutta methods will meet the so-called Butcher barrier [6], which means that the number of stages will be greater

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than the order of accuracy (e.g., the fifth order Runge–Kutta method requires six stages). On the other hand, the third order Runge–Kutta method will also lead to accuracy barrier, for example, the fifth order WENO scheme, coupled with the third order TVD Runge–Kutta method, can only achieve third order accuracy [34]. To obtain the ideal convergence rate, one needs to reduce the CFL condition number, this will increase the computational costs.

At the first decade of this century, many researchers made great efforts to overcome the aforementioned drawbacks. The appearance of ADER (Arbitrary DErivative in space and time) marks significant progress in this research field. It is a one-step and fully discrete Godunov approach of arbitrary high order accuracy in both space and time.

The idea of the ADER scheme can go back to the GRP (General Riemann Problem) scheme, which solves the generalized Riemann problem with piecewise smooth initial data instead of piecewise constant initial data (for the traditional Riemann problem). This generalized Riemann problem is often denoted by GRPR, with k the order of the approximation of the solution. One way to build a second order Godunov scheme is to solve the GRP1 only at interfaces of computational cells. The first nonlinear scheme based on such idea is constructed by Kolgan [16–18], and then further developed by van Leer and other researchers, see [42,43,32,9] for example. The first second order GRP scheme was proposed by Ben-Artzi and Falcozvitz in a manner different from the previous schemes: they evaluated the numerical fluxes by analytically solving a GRP1 at each cell interface. Among these schemes, Ben-Artzi and Falcozvitz’s work has attracted considerable attentions and has been well developed up to third order accuracy [4,3,5,19,48]. The GRP scheme is sophisticated, but may become quite complicated in the case of higher than third order accuracy. The ADER scheme simplifies the computational process of the original GRP scheme and makes it possible to construct schemes of arbitrary high order accuracy both in space and time in practice.

As pointed out by van Leer [46], the pioneering work of GRP may owe to Kolgan who introduced a second order GRP scheme by solving a generalized Riemann problem with piecewise linear initial data.

The first ADER is reported by Toro et al. in [33] for linear problems. Titarev and Toro extend the ADER to nonlinear equations [34]. Further developments of Toro, Munz, Dumbser and collaborators make ADER become a powerful high order scheme in various branches of CFD [34,38–38,27,28,10–12]. For more introduction of ADER’s extensions and applications, we refer the reader to Toro’s classical book [39, Chapter 20].

Comparing to the Godunov approach, Flux Vector Splitting (FVS) approach needs lower computational cost and reduces algorithm complexity, although most of the traditional FVS schemes, including the works of Steger–Warming [30], Van Leer [45], Zha–Bilgen[49] and Liu [24–26], often introduce over diffusion, especially near the intermediate fields. Recently, the TV scheme proposed by Toro and Vazquez reduces the numerical diffusion significantly [41]. This FVS scheme is also extended to high order accuracy based on the ADER framework [40,2]. Thus, the ADER is no more limited to a Godunov approach.

In general, the ADER scheme can be divided into two camps, one is the state-expansion [34], another one is the flux-expansion [36]. The former can not be integrated explicitly in time, so a Gaussian quadrature is necessary for high order accuracy. On the other hand, the latter can be integrated exactly in time, the time derivatives of the numerical flux, however, is required to be given explicitly, and it could be very tedious when the formulation of the numerical flux is complex. If one chooses a nonlinear Riemann solver such as HLIC, it is not trivial to deduce the formulation of high order derivative terms.

Besides the ADER scheme, HGKS (high-order accurate gas-kinetic scheme) is another high order accuracy scheme [20,21], which is a gas kinetic scheme. HGKS is based on the BGK model, for which the time derivatives in the Taylor expansion of the particle distribution function can be replaced by space derivatives directly. Compared with the ADER scheme, the numerical flux in HGKS can be obtained without a linearization process. HGKS can be also integrated exactly in time and made to be of arbitrary high order accuracy both in space and in time. We should point out, however, that the formulation of HGKS may also become very complex in the high order accuracy case. Although the recursive techniques simplify the procedure of deduction in the high order HGKS, the computational costs can not be reduced. In fact, the computational costs in CPU time for HGKS may be greater than those for WENO with a Godunov Riemann solver, see [22]. Another drawback for HGKS is that there are some spurious velocity and pressure oscillations near the contact discontinuity when HGKS is used to solve the Euler equations. This phenomenon is found and carefully analyzed by the authors in [7]. A remedy for the GKs coupled with the Runge–Kutta method is also provided in the articles [7,8]. However, it still remains unsolved how to construct an oscillation-free HGKS (of greater than second order accuracy).

In the last fifteen years, the ADER scheme and HGKS have provided new approaches to construct schemes of high order accuracy in both space and time. In this article, we present a new simple approach to construct a numerical scheme of arbitrary high order accuracy in both space and time.

Our proposed scheme will be based on the FVS-approach, but very different from the extension of the FVS-based framework of ADER. Comparing to the ADER scheme and HGKS, our new scheme provides a simpler version of high order scheme in both space and time which is easier to implement. We call the new scheme HFVS (arbitrary High order approach based on Flux Vector Splitting). We shall present a number of numerical tests to demonstrate the efficiency and accuracy of HFVS.

This paper is organized as follows. In Section 2, we present the framework of numerical schemes of arbitrary high order accuracy in both space and time, while in Section 3 we give the construction of our new scheme HFVS in details. In Section 4 we test HFVS in accuracy by a number of numerical examples of linear and nonlinear conservative laws in 1D and 2D, and compare the numerical results computed by HFVS and those by WENO in both accuracy and computational costs. Finally, we draw some conclusions.
2. Framework of numerical schemes of high order accuracy in both space and time

As mentioned in the previous section, the ADER scheme is a Godunov scheme based approach, while HGKS is a BGK model based approach. In our opinion, the flux-expansion version of the ADER scheme and HGKS can be put in a unified framework of high order schemes in both space and time.

Let us start with a one-dimensional hyperbolic system of conservation laws in the form:

\[
\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial F(W)}{\partial x} = 0.
\]  

(2.1)

To simplify the presentation, we consider here the 1D Euler equations of compressible flows in this section, although both the ADER scheme and HGKS are constructed for general hyperbolic conservation laws. For the 1D Euler equations, \(\mathbf{W}\) and \(F\) in (2.1) are given by

\[
\mathbf{W} = (\rho, \rho U, \rho E)^T, \\
F(W) = (\rho U, \rho U^2 + P, \rho EU + PU)^T,
\]

(2.2)

where \(\rho, U, E, P\) are the density, fluid velocity, specific total energy and pressure, respectively. The system can be closed by adding the equations of state (e.g., perfect gases):

\[P = (\gamma - 1)\rho \left( E - \frac{1}{2} U^2 \right).\]

Integrating (2.1) over a space–time computational cell \([x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}] \times [t^n, t^{n+1}]\), we have

\[
\int_{t^n}^{t^{n+1}} \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} \left( \frac{\partial \mathbf{W}}{\partial t} + \frac{\partial F(W)}{\partial x} \right) dx dt = 0.
\]

(2.3)

Denote \(\Delta x = [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}]\) and \(\Delta t = [t^n, t^{n+1}]\), we approximate (2.3) by the classical finite volume method (FVM)

\[
\mathbf{W}_j^{n+1} = \mathbf{W}_j^n - \frac{1}{\Delta x} \int_{t^n}^{t^{n+1}} \left( \mathbf{F}(j+\frac{1}{2}) - \mathbf{F}(j-\frac{1}{2}) \right) dt,
\]

(2.4)

where \(\mathbf{W}_j^n\) is the cell average of \(\mathbf{W}(x, t^n)\) \((x \in [x_{j-\frac{1}{2}}, x_{j+\frac{1}{2}}])\) and \(\mathbf{F}(j+\frac{1}{2})\) is the numerical flux. To make the scheme (2.4) high order accuracy, the key point is to suitably construct the numerical flux which should be in high order accuracy.

The procedure to construct a numerical scheme with high order accuracy in both space and time can be divided into three steps:

**Step I.** Calculate the space derivatives by certain reconstruction techniques.

**Step II.** Construct \(\mathbf{F}(j+\frac{1}{2})(0^+)^\ast\) by the Taylor expansion of the numerical flux as follows:

\[
\mathbf{F}(j+\frac{1}{2})(0^+)^\ast = \mathbf{F}(j+\frac{1}{2})(0^+) + \sum_{k=1}^{N} \frac{\partial^k}{\partial t^k} \left[ \mathbf{F}(j+\frac{1}{2})(0^+) \right] \frac{t^k}{k!}.
\]

(2.5)

**Step III.** Replace the time derivatives by spatial derivatives.

In the above procedure, a WENO reconstruction technique is used in Step I for the ADER scheme and HGKS, while very different techniques are employed in Steps II and III, and will be described in the following.

2.1. The ADER scheme

In the ADER scheme, an approximate Riemann solver such as HLLC is used in Step II:

\[
\mathbf{F}(j+\frac{1}{2})(0^+) = \begin{cases} 
\mathbf{F}(j), & \text{if } 0 \leq S_j, \\
\mathbf{F}(j) + S_j \left( \mathbf{W}_j^* - \mathbf{W}_j \right), & \text{if } S_j \leq 0 \leq S^*, \\
\mathbf{F}(j+1) + S_{j+1} \left( \mathbf{W}_j^* - \mathbf{W}_j \right), & \text{if } S^* \leq 0 \leq S_{j+1}, \\
\mathbf{F}(j+1), & \text{if } 0 \geq S_j,
\end{cases}
\]
where

\[ \mathcal{W}_k^\ast = \rho_k \left( \frac{S_k - U_k}{S_k - U^\ast} \right) \left[ \begin{array}{c} \frac{1}{S^\ast} \\ \frac{E_k + (S^\ast - U_k)(S^\ast + \frac{\rho_k}{\rho(S^\ast - U_k)})}{S_k - U^\ast} \end{array} \right], \quad k = j \text{ or } j + 1. \]

The wave speed of \( S_j \), \( S_j + 1 \) and \( S^\ast \) can have many choices, see [35].

Then, the time derivatives of numerical flux in (2.5) can be given analytically with the help of (2.2), for example, as follows.

\[
\frac{\partial}{\partial t} \mathcal{F}^{\ast} = \left[ (\rho U)_t, (\rho U)_x, U + (\rho U)_t, U_t + (\rho E + P), U_t (\rho E + P) + U (\rho E + P)_t \right]^T,
\]

\[
\frac{\partial^2}{\partial t^2} \mathcal{F}^{\ast} = \left[ (\rho U)_tt, (\rho U)_{tt}, U_{tt} + 2(\rho U)_x, U_{tt} + (\rho U)_t, U_{tt} (\rho E + P) \right. \\
\left. + 2U_t (\rho E + P)_t + U (\rho E + P)_{tt} \right]^T.
\]

According to the construction procedure of the Lax-Wendroff scheme, the time derivatives in the above formulations can be replaced by spatial derivatives.

**Remark 1.** The procedure can be performed for any given order. However, it is easy to see that these formulations may become very complicated in the case of very high order accuracy, thus leading to an increase of complexity in implementation.

### 2.2. HGKS

Different from the ADER scheme, HGKS is based on the BGK model:

\[ f_t + u f_x = \frac{g - f}{\tau}, \]

where \( f \) is the particle velocity distribution function and \( g \) is the corresponding equilibrium distribution. Both \( f \) and \( g \) are functions of space \( x \), time \( t \), the particle velocity \( u \), and the internal variables \( \xi = (\xi_1, \xi_2, \ldots, \xi_K) \) of \( K \) degrees of freedom.

The relationship between the numerical flux and the particle distribution function is

\[
\mathcal{F}^{j+\frac{1}{2}}(\tau) = \int u \Psi f^{j+\frac{1}{2}}(\tau) d\mathcal{B},
\]

where \( \Psi \) is the moment vector

\[ \Psi = (1, u, \frac{1}{2}(u^2 + \xi^2))^T, \]

and the internal variable \( \xi^2 \) is equal to \( \xi^2 = \xi_1^2 + \xi_2^2 + \cdots + \xi_K^2 \).

Hence, the time derivatives of numerical flux can be calculated by

\[ \frac{\partial}{\partial t} \mathcal{F}^{j+\frac{1}{2}} = \int u \Psi \frac{\partial f^{j+\frac{1}{2}}}{\partial t} d\mathcal{B}. \]

In view of the BGK model, the time derivatives can be evaluated directly, for example,

\[ f^{j+1}(0^+) = (1 - \tau (a^2 u + A^2))H(u)g^j + (1 - \tau (a^2 u + A^2))(1 - H(u))g^\prime. \]

\[
\frac{\partial}{\partial t} f^{j+1}(0^+) = \frac{1}{\tau} g_0 - \tau (a^2 + b) u^2 g_0 - \tau (A^2 + B^2) g_0 - 2 \tau (C + aA) u g_0 \\
+ [-\frac{1}{\tau} + A^2 + \tau (C^2 + aA) u + \tau ((a^2 + b) u^2)] H(u) g^j \]

\[ + [-\frac{1}{\tau} + A^2 + \tau (C^2 + aA) u + \tau ((a^2 + b) u^2)] (1 - H(u)) g^\prime, \]

\[
\frac{\partial^2}{\partial t^2} f^{j+1}(0^+) = -\frac{1}{\tau^2} g_0 - \frac{1}{\tau} u g_0 + \frac{1}{\tau} A g_0 + (a^2 + b) u^2 g_0 + (A^2 + B^2) g_0 + 2 (C + aA) u g_0.
\]
\[
+ \left[ \frac{1}{\tau^2} - \frac{1}{\tau} (a'd' + A') \right] + \frac{2}{\tau} \frac{d'd'}{\tau} - 2(C' + A')u - ((d')^2 + b')u^2 \right] \bar{H}(u) g' \\
+ \left[ \frac{1}{\tau^2} - \frac{1}{\tau} (a'd' + A') \right] + \frac{2}{\tau} \frac{d'd'}{\tau} - 2(C' + A')u - ((d')^2 + b')u^2 \right] (1 - \bar{H}(u)) g'.
\]

Here \( g_0 \) is the Maxwellian distribution at the cell interface \( x_{j+1/2} \), \( g' \) are the left and right limits of the Maxwellian distribution at the cell interface, respectively. \( a, b, A, \ldots \) are related to the space and time derivatives of the Maxwellian distribution, see [21] for the detailed definition.

**Remark 2.** Theoretically, HGKS can achieve arbitrary high order accuracy. However, due to its complexity in the expression, only third and fourth order schemes are often used in simulations [20,21].

In the following section, we want to present a simpler high order scheme, which is different from the ADER scheme and HGKS in construction.

3. **HFVS: arbitrary high order flux vector splitting scheme**

From the previous section, we can see that the complexity of both flux expansion version of the ADER scheme and HGKS actually comes from the procedure of evaluating the time derivatives of the numerical fluxes. So, if one can find a numerical flux, the time derivatives of which can be calculated very easily, then the complexity of the associated numerical scheme can be largely reduced. This motivates us to construct our scheme.

As is well known, in comparison with the traditional Godunov scheme, flux vector splitting (FVS) schemes are simpler in construction and coding. Thus, we just try to use the FVS approach, instead of the Godunov approach used in the ADER scheme. Our proposed scheme is constructed in the same framework as in Section 2, and also consists of three phases: constructing spatial derivatives, leading terms and time derivatives.

3.1. **Spatial derivatives**

In order to diminish possible spurious oscillations, we should use conservative reconstruction techniques such as ENO/WENO to evaluate spatial derivatives.

Suppose that a smooth function over a cell should be reconstructed, which is of the form:

\[
\bar{W}(x) = \bar{W}_j + \sum_{k=1}^{N} \hat{a}^k \frac{\partial^{\hat{W}} \phi_k(x)}{\partial x^k}, \quad x \in [x_{j-1/2}, x_{j+1/2}].
\]  \tag{3.6}

where \( \bar{W}_j \) is the known cell averaged value of \( \bar{W} \) over cell \( j \), \( \hat{a}^k \frac{\partial^{\hat{W}}}{\partial x^k} \) are the unknowns to be determined, and

\[
\varphi_1 = (x - x_j - 1) \Delta x, \quad \varphi_2 = \frac{1}{2} \left( \frac{x - x_j}{\Delta x} \right)^2 - \frac{1}{12}, \quad \varphi_3 = \frac{1}{6} \left( \frac{x - x_j}{\Delta x} \right)^3, \quad \varphi_4 = \frac{1}{24} \left( \frac{x - x_j}{\Delta x} \right)^4 - \frac{1}{80}.
\]

In general, we can always obtain two values on cell interfaces by the ENO/WENO reconstruction techniques, i.e., \( \bar{W}_j \pm 1/2 \).

Thus, we immediately obtain two linear algebraic equations:

\[
\begin{align*}
\bar{W}(x_{j+1/2}) &= \bar{W}_{j+1/2}, \\
\bar{W}(x_{j-1/2}) &= \bar{W}_{j-1/2}.
\end{align*}
\]  \tag{3.7}

In this case, these formulations also imply

\[
\frac{1}{\Delta x_i} \int_{i_j} \bar{W}(x) dx = \bar{W}_j.
\]

When the dimension \( N = 1 \), the above system is uniquely solvable.

For \( N > 1 \), in order to have a solution of the system (3.7), supplementary information should be added. A simple way of giving the supplementary information is to employ the cell averaged values. For example, for \( N = 2 \) we use

\[
\frac{1}{\Delta x_i} \int_{i_j} \bar{W}(x) dx = \bar{W}_j.
\]  \tag{3.8}
Coupling (3.8) with (3.7), we obtain a system of three independent linear equations for the three unknowns, which is uniquely solvable.

Similarly, for $N = 4$ we use

$$
\frac{1}{\Delta x} \int_{l_i} W(x) dx = W_l, \quad l = j - 1, j, j + 1,
$$

instead of (3.8). Obviously, this choice is appropriate for the case of arbitrary high order accuracy.

In fact, the spatial derivatives can be given explicitly, see below:

**Second order:**

$$
\left( \frac{\partial \tilde{W}}{\partial x} \right)_j = \tilde{W}_{j+\frac{1}{2}} - \tilde{W}_{j-\frac{1}{2}}. \quad (3.9)
$$

**Third order:**

$$
\left( \frac{\partial \tilde{W}}{\partial x} \right)_j = \tilde{W}_{j+\frac{1}{2}} - \tilde{W}_{j-1},
\left( \frac{\partial^2 \tilde{W}}{\partial x^2} \right)_j = 3(-2 \tilde{W}_{j+1} + \tilde{W}_{j+\frac{1}{2}} + \tilde{W}_{j-\frac{1}{2}}). \quad (3.10)
$$

**Fifth order:**

$$
\left( \frac{\partial \tilde{W}}{\partial x} \right)_j = \frac{1}{8}(\tilde{W}_{j-1} - \tilde{W}_j - 10 \tilde{W}_{j-\frac{1}{2}} + 10 \tilde{W}_{j+\frac{1}{2}}),
\left( \frac{\partial^2 \tilde{W}}{\partial x^2} \right)_j = \frac{1}{8}(-\tilde{W}_{j-1} - 50 \tilde{W}_j - \tilde{W}_{j+1} + 30 \tilde{W}_{j+\frac{1}{2}} + 30 \tilde{W}_{j-\frac{1}{2}}),
\left( \frac{\partial^3 \tilde{W}}{\partial x^3} \right)_j = \frac{1}{2}(-\tilde{W}_{j-1} + \tilde{W}_{j+1} - 2 \tilde{W}_{j+\frac{1}{2}} + 2 \tilde{W}_{j-\frac{1}{2}}),
\left( \frac{\partial^4 \tilde{W}}{\partial x^4} \right)_j = \frac{5}{12}(\tilde{W}_{j-1} + 10 \tilde{W}_j + \tilde{W}_{j+1} - 6 \tilde{W}_{j+\frac{1}{2}} - 6 \tilde{W}_{j-\frac{1}{2}}). \quad (3.11)
$$

### 3.2. Leading terms

Following the idea of FVS, the flux in a computational cell can be decomposed into two parts:

$$
\tilde{F}_j = \tilde{F}_j^+ + \tilde{F}_j^-.
$$

Thus, the numerical flux on a cell edge consists of two parts:

$$
\tilde{F}_{j+\frac{1}{2}}(0^+) = \tilde{F}_{j+1}(0^+) + \tilde{F}_{j+\frac{1}{2}}(0^+). \quad (3.12)
$$

By a first-order Steger–Warming scheme [30], we have

$$
\tilde{F}_j^\pm \left( \tilde{W} \right) = \Lambda^\pm \tilde{W}, \quad (3.13)
$$

where $\Lambda^\pm = L \Lambda^\pm \Lambda$, $L$ and $R$ are the left and right eigenvectors respectively, $\Lambda^\pm$ are the diagonal matrices of positive resp. negative eigenvalues. On the other hand, we also have

$$
\tilde{F}_j^\pm \left( \tilde{W} \right) = \tilde{A}^\pm \tilde{W}, \quad (3.14)
$$

where $\tilde{A} = \tilde{L} \Lambda \tilde{R}$ is the Jacobian matrix, $\Lambda = \Lambda^+ + \Lambda^-$ is the diagonal matrix of eigenvalues.
3.3. Time derivatives

By virtue of (3.13), (2.1) and (3.14), omitting the subscripts, we obtain the time derivatives of two parts in (3.12):

$$\frac{\partial}{\partial t} F = A^\pm \frac{\partial}{\partial t} W = A^\pm \left( -A \frac{\partial}{\partial x} W \right).$$

We should point out here that in the above identity we have assumed that $A^\pm$ do not depend on $t$ locally in space and time (i.e., within a cell and from $t^n$ to $t^{n+1}$). This is reasonable from the numerical point of view (also cf. [34,36]).

Generally, we have

$$\frac{\partial^k}{\partial t^k} F = \frac{\partial^{k-1}}{\partial t^{k-1}} F = \frac{\partial^{k-1}}{\partial t^{k-1}} \left( A^\pm \left( -A \frac{\partial}{\partial x} W \right) \right) = A^\pm (-A)^k \frac{\partial^k}{\partial x^k} W.$$

Hence, we can easily get the numerical flux (in high order accuracy) as follows:

$$\tilde{F}_{j+\frac{1}{2}} (\tau) = \tilde{F}_{j+\frac{1}{2}} (0^+) + A^\pm \sum_{k=1}^{N} (-A)^k \frac{\partial^k}{\partial x^k} W \frac{\tau^k}{k!}.$$  \hspace{1cm} (3.16)

where the spatial derivatives $\frac{\partial^k}{\partial x^k}$ $(k = 1, \cdots, N)$ are given in (3.9)–(3.11).

Remark 3. Compared with the flux-expansion version of the ADER scheme and the HGKS scheme, the current scheme uses the Steger–Warner flux. Moreover, Riemann solver, to evaluate the leading term. Furthermore, the high order terms in the Lax–Wendroff procedure also become simpler by freezing the eigen-matrices in the Steger–Warner solver. And since the current scheme can be written in a simple and compact formulation (3.16), which is simpler than that of the original flux-expansion version of ADER (see Section 2.1) and HGKS (see Section 2.2), the current scheme is easier to implement.

Remark 4. Although the Steger–Warner scheme is used as the leading term in the construction of our scheme, we should point out that the leading term in (3.16) can be any exact/approximate Riemann solver, such as the HLLC and BGK solvers, while the rest terms can keep unchanged. This means that the latter terms in (3.16) can be regarded as a building block, in the sense that they can be coupled with any exact/approximate Riemann solvers as the leading term to extend a first order scheme to high order accuracy in both space and time.

4. Numerical tests

In this section, we present the numerical experiment results for linear and nonlinear hyperbolic conservative laws in 1D and 2D. As aforementioned, we call our new scheme the HFV (An Arbitrary High Order Approach Based On Flux Vector Splitting) scheme. The abbreviations HFV52, HFV3 and HFV35 mean the second-, third- and fifth-order HFV schemes, respectively. For simplicity, we only use conservative variables ENO/WENO reconstruction techniques for all numerical schemes. And we shall find that they perform well in the most cases below.

4.1. Linear hyperbolic problems

In this subsection, we want to examine the accuracy of the new scheme by solving some linear hyperbolic problems.

**Example 1 (Accuracy test).** We consider the following linear convection equation

$$\frac{\partial w}{\partial t} + \frac{\partial w}{\partial x} = 0$$

with initial data

$$w_0(x) = \sin(2\pi x).$$

We take the unit interval $(0, 1)$ as the computational region and prescribe suitable boundary conditions (periodic boundary condition).

Table 1 shows the errors and accuracy for different schemes using the same CFL number 0.95 at a computational time $t = 1.0$. We can observe that the HFV scheme can reach its designed order accuracy.

4.2. Nonlinear hyperbolic problem

In this section, we proceed to present the numerical results of 1D and 2D Euler equations of compressible flows.
Table 1

<table>
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<tr>
<th>Method</th>
<th>$N$</th>
<th>$L_1$ error</th>
<th>Order</th>
<th>$L_2$ error</th>
<th>Order</th>
<th>$L_\infty$ error</th>
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Fig. 1. Example 2. Density profiles computed by the HFVS schemes of different order. From left to right are results of second order, third order and fifth order schemes, respectively.

4.2.1 1D Euler equations

In this section we test our scheme for the 1D Euler equations (2.2). We take $\gamma = 1.4$ and the CFL number $= 0.95$ for all the numerical schemes.

Example 2 (Shu–Osher shock–turbulence interaction problem [29]). We take $(-1, 1)$ to be the computational domain and

$$
(\rho, u, P) = \begin{cases} 
(3.857143, 2.629369, 10.333333), & x < -0.8, \\
(1 + 0.2 \sin 5\pi x, 0.0, 1.0), & x > -0.8 
\end{cases}
$$

to be the initial data.

Figs. 1 and 2 present the density and internal energy profiles computed by the HFVS schemes of different order at $t = 0.47$ on a mesh of 200 cells. A reference solution is obtained by using the third order WENO scheme on a fine mesh of 2000 cells. We can observe that all the schemes can resolve the complex solution of the governing equations. In addition, HFVS2 introduces over diffusion and HFVS3 improves the resolution, while HPVS5 gives a sharp profile. This also demonstrates the advantages of high order schemes.

Example 3 (Woodward–Colella blast wave problem [47]). The computational domain is $[0, 1]$ with reflected boundary conditions on both sides, and the initial data are

$$
(\rho, U, P) = \begin{cases} 
(1.0, 0, 1000), & 0 \leq x \leq 0.1, \\
(1.0, 0.01), & 0.1 \leq x \leq 0.9, \\
(1.0, 100), & 0.9 \leq x \leq 1.0. 
\end{cases}
$$
This is a problem with very high ratio of the pressure, the solution of which includes strong shock waves, contact discontinuities, rarefaction waves and their interactions. So it is a very challenging problem for high order schemes.

In Figs. 3 and 4 the numerical results on a mesh of 800 cells at time $t = 0.038$ are presented. The reference solution is obtained by using the third order WENO on a mesh of 10000 cells. All the HFVS schemes can resolve all the waves of the flow. As in the previous test problem, HFVS2 gives a less resolved solution, while HFVS3 greatly improves the resolution, and HFVS5 resolves very sharply. These numerical results demonstrate the robustness and accuracy of the HFVS scheme.

4.2.2 2D nonlinear problems

In this section, we present the numerical results of the 2D Euler equations:

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}(\mathbf{W})}{\partial x} + \frac{\partial \mathbf{G}(\mathbf{W})}{\partial x} = 0$$
where \( W = (\rho, \rho U, \rho V, \rho E)^T \), and

\[
\begin{align*}
\overline{F}(W) &= (\rho U, \rho U^2 + P, \rho UV, \rho EU + PU)^T, \\
\overline{G}(W) &= (\rho V, \rho UV, \rho V^2 + P, \rho EV + PV)^T,
\end{align*}
\]

and the equation of state is

\[
P = (\gamma - 1) \rho \left( E - \frac{1}{2} U^2 - \frac{1}{2} V^2 \right).
\]

In the following simulations we take \( \gamma = 1.4 \) and the CFL number = 0.45 for all the numerical schemes.

**Example 4 (2D Riemann problem).** We want to show two cases which are simple extensions of 1D Riemann problems.

In the first case we take the computational domain to be \([0, 1] \times [0, 1]\), and the boundary conditions to be outflow condition. The initial data are given by

\[
(\rho, U, V, P) = \begin{cases} 
(1.0, 0.0, 0.0, 1.0), & x < 0.5 \text{ and } y < 0.5, \\
(0.1, 0.0, 0.0, 0.1), & x < 0.5 \text{ and } y > 0.5, \\
(1.0, 0.0, 0.0, 1.0), & x > 0.5 \text{ and } y > 0.5, \\
(0.1, 0.0, 0.0, 0.1), & x > 0.5 \text{ and } y < 0.5.
\end{cases}
\]

We proceed to test the accuracy of the HFVS schemes of different order. The numerical results using a mesh of \( 100 \times 100 \) cells at \( t = 0.15 \) are presented in Fig. 5, which shows the density contours between the range of 0.1 and 1.0. We can draw the same conclusions as in the 1D case, i.e., all the schemes of different order work well, and the accuracy and resolution can be improved if the order of the scheme is increased.

In the second case we test our scheme for interaction of different waves and boundaries. The computational domain is \([0.1] \times [0.1]\) and the boundary conditions are all reflective conditions. The initial data are given by

\[
(\rho, U, V, P) = \begin{cases} 
(1.0, 0.0, 1.0), & \sqrt{(x - 0.5)^2 + (y - 0.5)^2} \leq 0.3, \\
(0.125, 0.0, 0.1), & \text{else}.
\end{cases}
\]

The numerical results using a mesh of \( 100 \times 100 \) cells at \( t = 1.0 \) are presented in Fig. 6 which gives the density contours between the range of 0.1 and 0.6. Remark that at \( t = 1.0 \) the shock waves do hit the boundary and the reflected waves interact each other, thus resulting in a complex pattern of the flow. From Fig. 6 we can draw the same conclusion as in the previous test, i.e., the second order scheme results in a over-diffusive solution, while the third order scheme improves the accuracy, and the fifth order scheme gives a sharp profile. This also demonstrates high accuracy of high order schemes.

**Example 5 (Double Mach reflection problem).** This is a widely used 2D benchmark problem proposed by Woodward and Colella.

The computational domain is \([0, 3] \times [0, 1]\), and the boundary data are given below.

Left boundary: inflow;
Right boundary: outflow;
Top boundary: If $x < \frac{1}{6} + \frac{\sqrt{2}}{4}t$, use inflow boundary condition, else use reflective boundary condition.
Bottom boundary: If $x > \frac{1}{6}$, use inflow boundary condition, else use reflective boundary condition.
The initial data are

$$
(\rho, \rho U, \rho V, \rho E)^T = \begin{cases} 
(8.0, 57.1597, -33.0012, 563.544), & y > \sqrt{3}(x - 1/6), \\
(1.4, 0.0, 0.0, 2.5), & \text{else}. 
\end{cases}
$$

The numerical results at $t = 0.2$ are presented and 50 contours are set between a range of 1 to 22.

Fig. 7 presents the numerical results of HFVS2, HFVS3 and HFVS5 using a mesh of 960 x 320 cells. It can be clearly seen that higher order scheme can give more details of this complex flow than lower one. This show that one should use a high order scheme in the simulations of complex flows.

4.3. Further discussions

4.3.1. Impact on resolution of different leading terms

As aforementioned, it seems possible to use different leading terms to incorporate the same high derivative terms in the HFVS scheme. To see this, we compare three solvers: Steger–Warming, HLLC and BGK, which represent the FVS, Godunov and GKS approaches. Fig. 8 shows the numerical results computed by the second order schemes for the Shu–Osher problem, while Fig. 9 gives the results of the fifth order schemes. We can observe that all the schemes using different leading terms perform well. And there are small differences among these schemes in the low order case, while these differences almost vanish in the high order case.

These numerical results do demonstrate that the high order derivatives terms of the HFVS scheme can be used as a building block to couple with any Riemann solvers.

4.3.2. Comparison between WENO and HFVS

To demonstrate the efficiency, we give the detailed comparison between the Runge–Kutta WENO scheme and the HFVS scheme. We should point out here that we use the same reconstruction step in both WENO and HFVS schemes in our simulation. We use WENO3 and WENO5 to represent the third order and fifth order WENO schemes respectively, and RK3 the third order Runge–Kutta method.

Table 2 presents the errors and accuracy of WENO3+RK3 and WENO5+RK3 using the same computational condition as in Table 1. From Table 2 we see that although it results in better accuracy than WENO3+RK3, WENO5+RK3 can only achieve the third order accuracy.

Fig. 11 shows the numerical results computed by the WENO and HFVS schemes of the Shu–Osher problem. Clearly, one observes that the HFVS scheme has better accuracy than WENO, no matter in the third or fifth order case.

Table 3 and Fig. 10 give comparisons of computational costs between the WENO and HFVS schemes. We can also see that the HFVS schemes use almost half CPU time of WENO schemes in one computational time step, and can achieve the same convergence error by using less CPU time. The reason is in our opinion that in each computational time step, the HFVS scheme requires only one reconstruction step, while WENO has to employ three reconstruction steps due to the use of the third order Runge–Kutta method. Regardless of the fact that the HFVS scheme should calculate the high order derivative terms, the evaluation of the high order derivative terms is still cheaper than reconstruction steps.

5. Conclusions and discussions

In this paper, based on the idea of flux vector splitting schemes, we have proposed a new scheme of arbitrary high order accuracy in both space and time for both linear and nonlinear hyperbolic conservative laws. We have followed the following
**Fig. 7.** Example 5. Numerical results of double Mach problem. The profiles from top to bottom are the results of HFVS2, HFVS3 and HFVS5, respectively.

**Fig. 8.** Shu–Osher problem. Density profiles for HFVS using different leading terms: left: global picture, right: local enlargement.
Table 2
Errors and accuracy for WENO.

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<th>$L_2$ error</th>
<th>Order</th>
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WENO5+RK3

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Table 3
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Fig. 9. Shu–Osher problem. Density profiles for HFVSS using different leading terms: left: global picture, right: local enlargement.

Fig. 10. Computational costs versus convergence errors.
strategy in the construction. We start from splitting all the space and time derivatives in the Taylor expansion of the numerical flux into two parts: one part with positive eigenvalues, another one with negative eigenvalues. Then, all the time derivatives are replaced by spatial derivatives, according to a Lax–Wendroff procedure. A state-of-art WENO reconstruction polynomial is used to calculate the space derivatives in order to eliminate possible spurious oscillations. Therefore, it could be an alternative for CFD softwares.

In addition, we have noticed that the high order (in time) derivatives terms in the Taylor expansion of the numerical flux can be used as a building block, in the sense that they can be coupled with any (Riemann) solvers to extend a first-order scheme to very high order accuracy in both space and time.

Numerous numerical tests for linear and nonlinear hyperbolic conservative laws have been carried out to demonstrate robustness, high order accuracy in both space and time, and computational cheapness of the new scheme.

In this article, although only the Euler equations, a basic model in hydrodynamics, are considered, we believe the HFVS scheme can be extended to the numerical solution of more general problems. For example, it can be easily extended to the Radiation Hydrodynamical Equations (RHE), for which the related work is undergoing. Also, we plan to extend the HFVS scheme to the Navier–Stokes (NS) equations. It is well known that the traditional FVS schemes often introduce over diffusion, which may not give a satisfactory solution of the NS equations. In this article, however, we have seen that the HFVS scheme can extend any first order Riemann solver to achieve high order accuracy in both space and time. Similarly, we can choose the first order BGK scheme as a low order solver and follow the process in Section 3 to get a high order NS solver to diminish over diffusion. We shall verify this idea in the near future.

References


